A Brief Introduction to Error Analysis

Errors and Uncertainties

Suppose the theory you are testing predicts a value of 2.0 for a particular quantity, but your experimental value is 1.5. Is your experimental value consistent with the theory or not?

This sort of situation arises all the time with experiments: the measured number is not exactly identical with the predicted theoretical value. To draw any conclusion from the experiment, it's necessary to know what the precision of the measured number is. For example, if the measured number and its uncertainty are 1.5 $\pm$ 0.5, the measurement is consistent with the theoretical prediction of 2.0; however, if the measured number were 1.5 $\pm$ 0.1, it is not. Knowing the uncertainty in your measured number is an essential part of the experiment. The uncertainty number (the $\pm\,0.1$) is customarily called the "experimental error" although the term is somewhat misleading - no sort of error has necessarily been made by anyone - uncertainties are inherent in the measuring process due to the limitations of the instruments used and sometimes due to the statistical uncertainty inherent in the natural phenomenon itself.

Types of Errors

It is important to realize that uncertainties can get into an experiment in a surprising number of different ways, and often just identifying the major source of uncertainty can be a challenge. The most obvious way uncertainties enter is through the limitations of the measuring instruments; for example, the smallest markings on the ruler are millimeters, so the uncertainty of a ruler measurement must be at least a millimeter. However, the uncertainty of a ruler measurement could easily be much more than a millimeter. Suppose you are measuring the focal length of a lens by focusing light from a distant source on a screen and measuring the lens-to-screen distance with a ruler. Different distances (maybe differing by a few mm) might look equally in focus to you, so the uncertainty in this case is a few mm. This is a ***problem of definition*** – the visual determination of "in focus" has uncertainty.

As another example, you might try measuring the length of a tabletop and find that you get slightly different measurements depending on what part of the table you measure – because its sides are not quite smooth or perhaps imperfectly parallel. Here again the uncertainty could be larger than the uncertainty coming from the ruler's markings. This uncertainty is due to ***model error***: the model of the tabletop as rectangle is not exact.

Sometimes the way a measuring device is read contributes an error: if you are constrained to view the ruler from an angle rather than perpendicularly, the geometry of the viewing can result in ***parallax error***.

Another possibility is that the ruler was manufactured poorly, with the markings 1% too close together – then all measurements made with this ruler will be low by 1%. This non-random error due to miscalibrated equipment, which is always present to some degree, is called ***systematic error***. It can be difficult to estimate the size of systematic errors: sometimes the equipment manufacturer specifies a probable or guaranteed maximum degree of systematic uncertainty in an equipment manual, sometimes an experimenter must undertake a separate *calibration experiment* to test the measuring instrument against a known standard (which itself has a specified maximum uncertainty, hopefully small).

Some uncertainties are actually ***random***, and then the degree of uncertainty can be measured accurately by repeating the experiment many times – the degree of scatter among the measured values (i.e. standard deviation) can usually be taken as the measure of the uncertainty. In practice, however, uncertainties usually result from many sources – systematics, random uncertainties, and various model errors – and it is not always easy to figure out what source of uncertainty is dominant and which can be safely ignored.

As a last word on experimental error, it is important to understand that “human error” is **not** a legitimate type of experimental error. In other words, if you did a procedure wrong or wrote down a wrong number, **this does not count as “experimental error”** – it is simply a mistake. Note that *the lab can be made available day and night, so if you should discover a mistake, you are encouraged to return and redo a procedure or an experiment.*

Estimating Uncertainties

As the above discussion indicates, figuring out the size of an experimental uncertainty can be tricky. Fortunately, we usually only require an estimate, and this is usually not too difficult. For ruler and other scale reading measurements, $\pm$ half the smallest scale division is a reasonable estimate for the uncertainty (but be alert for situations where the actual uncertainty is larger, as in the examples above). This rough estimate will handle ruler and balance measurements. For timing measurements (including velocity), it is often convenient to repeat the measurement a few times - the maximum deviation from average gives a rough, order-of-magnitude measure of the uncertainty.

Propagating Uncertainties

Once you know the uncertainties in the raw measured quantities in an experiment, you may still need to know the uncertainty in some other value calculated from the raw quantities. For example, after you measure the mass $m \pm \Delta m$ and the volume $V \pm \Delta V$ of an object, you might want to know the mass density and its uncertainty. Of course, the mass density ρ is given by $\rho = \frac{m}{V}$ – but what is its uncertainty? This is the problem of *propagation of uncertainties* – figuring how uncertainties are affected when they propagate through a calculation or a series of calculations. There are well-established statistical rules for how to figure this out – but the rules are a little complicated and actually only rigorously valid for random uncertainties following the normal, or bell-curve, frequency distribution (although the method is *approximately* valid generally, and in practice used almost universally). We present instead a simpler approach for estimating the uncertainties in this course (which are usually dominated by systematic rather than random uncertainties): the uncertainty $\Delta\rho$ in the density is given by $\Delta\rho = \frac{1}{2}(\rho_{\max} - \rho_{\min})$, where $\rho_{\max}$ is the maximum density consistent with the data and $\rho_{\min}$ is the minimum density. Hence we have $\rho_{\max} = \frac{m+\Delta m}{V-\Delta V}$ and $\rho_{\min} = \frac{m-\Delta m}{V+\Delta V}$. Take a close look at the plus and minus signs in these relations and note that $\rho_{\max}$ is *not* the maximum mass over the maximum volume, but rather the maximum mass over the *minimum* volume.

The above approach to propagating uncertainties gives a *worst-case estimate*: the density reaches its limiting value $\rho \pm \Delta\rho$ only when *both* mass and volume are at the (appropriate) ends of their respective ranges. If the uncertainties in mass and volume were random and uncorrelated, one might expect this event (both mass and volume simultaneously at the limit of their allowed ranges) to be rather unlikely, and the above procedure would then give an *overestimate* of the uncertainty. On the other hand, if the uncertainties in mass and volume are not random but due to systematic errors, we may in fact be most interested in this worst-case bound on the uncertainty. And as a practical matter, this estimate of the uncertainty will in most cases be not too much larger (say, within a factor of 2) than that calculated by the statistically rigorous approach. And in most cases, this degree of accuracy in the estimate of uncertainties is quite adequate – the calculation of uncertainties is sometimes not a very exact science! Thus, it is always important to document **how** you calculated your uncertainties.

Uncertainty in the Slope and y-intercept of a Linear Fit

(Reference: J. Higbie, Am. J. Phys., Vol. 59, No. 2, February 1991)

Though the derivation goes beyond the scope of this class, one can calculate the uncertainty in the slope and y-intercept from the correlation parameters given to you in your linear fit.

- The uncertainty in the slope (δm) is given by:

$$\delta m = \frac{|m| \tan(\arccos(R))}{\sqrt{N-2}} \tag{1}$$

  where m is the slope, R is the square root of the R^2 value from the linear fit, and N is the number of data points in the data set. Note that there cannot be any uncertainty in the fit of you data if there are no more than 2 data points!

- The uncertainty in the y-intercept (δb) is given by:

$$\delta b = \delta m \cdot x_{\mathrm{rms}} \tag{2}$$

  where x_{rms} is the root mean square value of the x values.

- The root mean square value of a set of x values can be found as

$$x_{\mathrm{rms}} = \sqrt{\frac{1}{N}\left(\sum_{i=1}^{N} x_i^2\right)} \tag{3}$$

  As you can see, it is the "root" of the "mean" of the "squares". Note that, the further the values are from the y-axis the larger the x_{rms} value and consequently the larger the uncertainty in the y-intercept for a given slope uncertainty.

Further Reading

Students interested in further reading on the subject of error analysis are directed to John Taylor's excellent and highly accessible text *An Introduction to Error Analysis.*